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Feynman Path Integrals in the Young Double-Slit Experiment

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An estimate for the value of the nonlinear interference term in the Young double-slit experiment is found using the Feynman path-integral method. In our time-dependent calculation the usual interference term becomes multiplied by $1+e$ with e proportional to $\cos(2m\lambda L/\hbar T)$, where λ is the distance between the two slits (holes) and L is the length of the shortest trajectory of electrons between the source and the observation point.

1. INTRODUCTION

The interference experiment done by Young with the experimental arrangement of Figure 1 clearly showed the wave nature of light. The same type of experiment with an electron beam indicates that electrons have the property of wave in accordance with the prediction of quantum mechanics: The wave function ψ for Figure 1a is the sum of the two wave functions ψ_1 and ψ_2 corresponding to the situation of Figures 1b and 1c, respectively,

$$\psi = \psi_1 + \psi_2 \tag{1}$$

This equality (1) is often stated as an example of the superposition principle of quantum mechanics.

Strictly speaking, however, the wave functions ψ, ψ_1, and ψ_2 are the solutions of the Schrödinger equation with different boundary conditions; they correspond to the experimental arrangements of Figures 1a, 1b, and 1c, respectively. Therefore, the wave functions ψ, ψ_1, and ψ_2 belong to different Hilbert spaces, and equation (1) does not hold in a rigorous sense.

Another way to see the problem with equation (1) is to apply the viewpoint of the Feynman path integral. In this picture we expect contributions from paths like that shown in Figure 2, but these are unlikely to be

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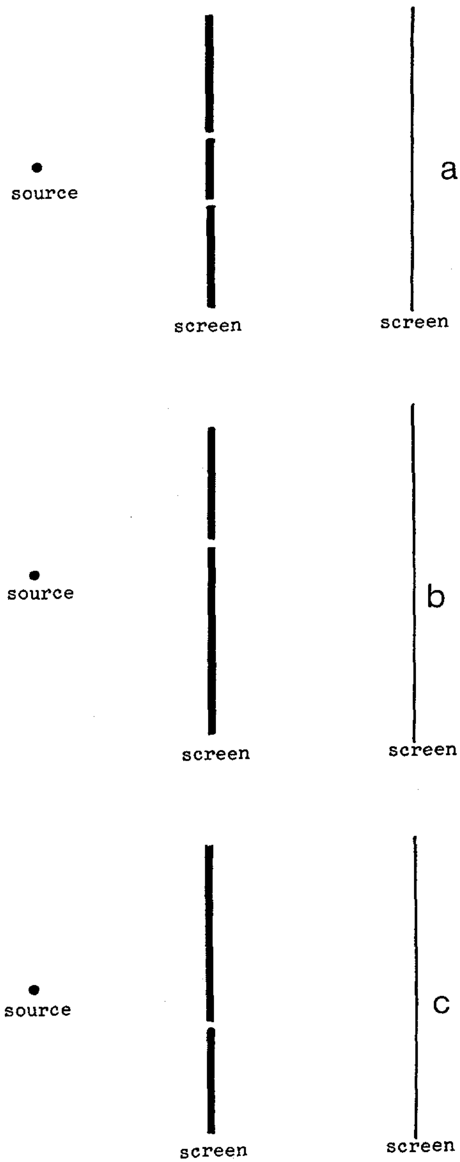

Fig. 1. Experimental arrangements.

included in equation (1). The contribution coming from the paths looping through the two slits are expected to be of relatively order $\hbar^n$ with $n \geq 1$. Therefore, equation (1) is indeed a very good approximation for the wave function ψ.

In the present paper we are going to calculate the quantity $\psi - \psi_1 - \psi_2$ in the Feynman path integral approach. In the next section we develop our calculational technique, and in Section 3 we apply it to the double-slit problem in a simplified two-dimensional case. Its generalization to the three-dimensional case is straightforward and is discussed in Section 4.

2. CALCULATION FOR TYPICAL COMPOSITIONS OF PATHS

Let us consider our problem in two-dimensional space with coordinates chosen as in Figure 2. The Lagrangian of our system is taken to be

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - V(x, y) \tag{2}$$

where

$$V(x, y) = \begin{cases} V & \text{for } 0 \leq x \leq \delta;\ y \leq -\frac{\lambda}{2} - \varepsilon,\ -\frac{\lambda}{2} \leq y \leq \frac{\lambda}{2},\ \frac{\lambda}{2} + \varepsilon \leq y \\ & \text{(the shaded area in Figure 2)} \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

We are going to approach the problem by applying the method of the Feynman path integral. In our case the particle moves freely except at the screens. Therefore, the key ingredient in our calculation is the free propagator $K(b, a)$ defined as

$$K(b, a) = \int_a^b [Dx] \exp \frac{i}{\hbar} S[b, a] \tag{4}$$

where $S[b, a]$ is the free action

$$S[b, a] = \int_{t_a}^{t_b} \frac{m}{2} \{\dot{x}(t)^2 + \dot{y}(t)^2\}\, dt \tag{5}$$

the symbol $[Dx]$ stands for the path integral measure, and $a = (t_a, A) = (t_a; x_a, y_a)$, etc. Explicitly we have (Feynman and Hibbs, 1965)

$$K(b, a) = \left[\frac{2\pi i\hbar(t_b - t_a)}{m}\right]^{-1} \exp \frac{im\{(x_b - x_a)^2 + (y_b - y_a)^2\}}{2\hbar(t_b - t_a)} \tag{6}$$

In order to estimate the contribution coming from paths such as that shown in Figure 2 it is necessary to have a method to calculate the transition amplitude for a composition of paths. For this purpose we make use of the

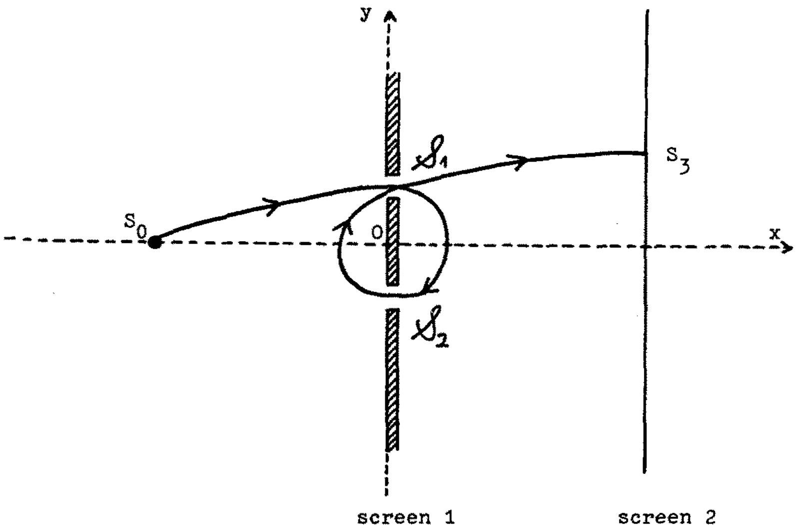

Fig. 2. A typical path looping through the two slits. The following notations will be used in the text: S_0 (source) $=(x_0, y_0)$; $\mathscr{S}_1$ (upper slit) $=\{(x_1, y_1)|0\leq x\leq \delta, \lambda/2\leq y\leq \lambda/2+\varepsilon\}$; $\mathscr{S}_2$ (lower slit) $=\{(x_2, y_2)|0\leq x\leq \delta, -\lambda/2-\varepsilon\leq y\leq -\lambda/2\}$; S_3 (observation point on screen 2) $=(x_3, y_3)$, $l=$ distance between S_0 and $\mathscr{S}_1$ (or $\mathscr{S}_2$); $\lambda=$ distance between $\mathscr{S}_1$ and $\mathscr{S}_2$; $l_1=$ distance between $\mathscr{S}_1$ and S_3; $l_2=$ distance between $\mathscr{S}_2$ and S_3; $L_1=l+l_1$; $L_2=l+l_2$.

following identity:

$$K(b, a)=\int K(b, c)K(c, a)\, dx_c\, dy_c \tag{7}$$

with t_c being fixed in the interval

$$t_b > t_c > t_a \tag{8}$$

By inserting a trivial equality

$$\frac{1}{t_b-t_a}=\int_{t_a}^{t_b} dt_c \tag{9}$$

into (7), we get

$$K(b, a)=\frac{1}{t_b-t_a}\int_{t_a}^{t_b} dt_c \int dx_c\, dy_c\, K(b, c)K(c, a) \tag{10}$$

When we are interested in calculating contributions from paths which go from A to B passing through a given small region $\mathscr{C}$ (Figure 3a), we get the corresponding transition amplitude by restricting the space integral in (10) to the domain $\mathscr{C}$:

$$K(b, a)_{\mathscr{C}}=\frac{1}{t_b-t_a}\int_{t_a}^{t_b} dt_c \int_{(x_c,y_c)\in\mathscr{C}} dx_c\, dy_c\, K(b, c)K(c, a) \tag{11}$$

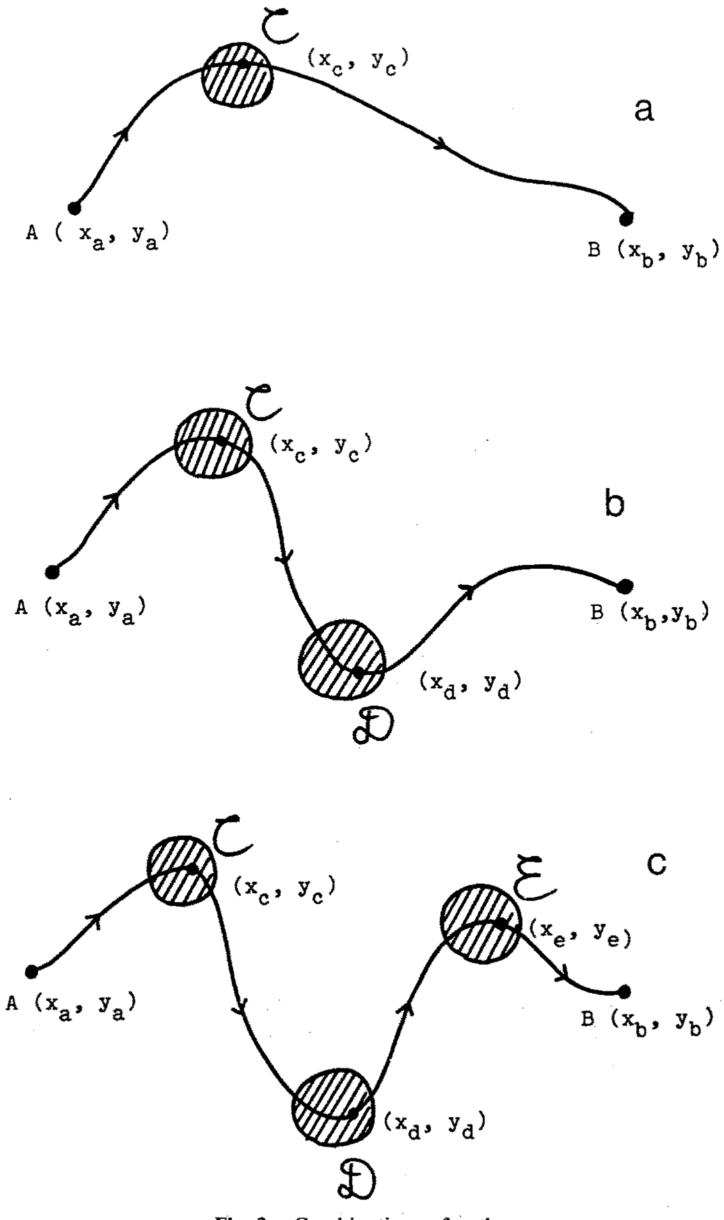

Fig. 3. Combinations of paths.

By changing the order of integration, the integral over t_c is estimated in Appendix 1 to be

$$I_1 = \frac{1}{t_b - t_a}\int_{t_a}^{t_b} dt_c\, K(b, c)K(c, a)$$

$$\approx \left(\frac{2\pi i\hbar}{m}\right)^{-3/2}(t_b - t_a)^{-3/2} l_{bc}^{-1/2} l_{ca}^{-1/2} \exp\left[i\frac{m(l_{bc} + l_{ca})^2}{2\hbar(t_b - t_a)}\right] \tag{12}$$

where

$$l_{bc} = [(x_b - x_c)^2 + (y_b - y_c)^2]^{1/2}$$

and

$$l_{ca} = [(x_c - x_a)^2 + (y_c - y_a)^2]^{1/2} \tag{13}$$

Since we consider only the case where the region $\mathscr{C}$ is small, the right-hand side of (11) can be written as

$$K(b, a)_{\mathscr{C}} \approx |\mathscr{C}| \cdot I_1 \tag{14}$$

where $|\mathscr{C}|$ is the area of the domain $\mathscr{C}$. In order for the approximation (14) to be valid, it is necessary that the phase of (12) does not vary much in the domain $\mathscr{C}$. Let us take two points C and C' in $\mathscr{C}$. Then the phase of (12) differs between these two points C and C' by

$$\frac{m}{2\hbar(t_b - t_a)}[(l'_{bc} + l'_{ca})^2 - (l_{bc} + l_{ca})^2]$$

$$= \frac{m}{2\hbar(t_b - t_a)}(l'_{bc} + l'_{ca} + l_{bc} + l_{ca})(l'_{bc} - l_{bc} + l'_{ca} - l_{ca})$$

$$\approx \frac{m}{\hbar(t_b - t_a)}(l_{bc} + l_{ca})(\Delta l_{bc} + \Delta l_{ca}) \tag{15}$$

Therefore, in (11) we should demand that variation of one of the two-dimensional (spatial) integration variables which is responsible for elongation or contraction of the path be restricted by

$$\frac{\hbar(t_b - t_a)}{m(l_{bc} + l_{ca})} \tag{16}$$

In this way, to have consistency of calculation, we are led to the restriction that

$$|\mathscr{C}| \lesssim O(\hbar) \tag{17}$$

In the same way we obtain in Appendices 2 and 3 the transition amplitudes corresponding to Figures 3b and 3c:

$$K(b,a)_{\mathscr{D}\mathscr{C}}=\frac{1}{t_b-t_a}\int_{t_a}^{t_b}dt_d\int_{(x_d,y_d)\in\mathscr{D}}dx_d\,dy_d\,K(b,d)K(d,a)_{\mathscr{C}}$$
$$\approx\left(\frac{2\pi i\hbar}{m}\right)^{-2}|\mathscr{C}|\,|\mathscr{D}|(t_b-t_a)^{-2}l_{bd}^{-1/2}l_{dc}^{-1/2}l_{ca}^{-1/2}$$
$$\times(l_{bd}+l_{dc}+l_{ca})^{-1/2}\exp\left[i\frac{m(l_{bd}+l_{dc}+l_{ca})^2}{2\hbar(t_b-t_a)}\right] \quad (18)$$

$$K(b,a)_{\mathscr{E}\mathscr{D}\mathscr{C}}=\frac{1}{t_b-t_a}\int_{t_a}^{t_b}dt_e\int_{(x_e,y_e)\in\mathscr{E}}dx_e\,dy_e\,K(b,e)K(e,a)_{\mathscr{D}\mathscr{C}}$$
$$\approx\left(\frac{2\pi i\hbar}{m}\right)^{-5/2}|\mathscr{C}|\,|\mathscr{D}|\,|\mathscr{E}|(t_b-t_a)^{-5/2}l_{be}^{-1/2}l_{ed}^{-1/2}l_{dc}^{-1/2}l_{ca}^{-1/2}$$
$$\times(l_{be}+l_{ed}+l_{dc}+l_{ca})^{-1}\exp\left[i\frac{m(l_{be}+l_{ed}+l_{dc}+l_{ca})^2}{2\hbar(t_b-t_a)}\right] \quad (19)$$

where we have for $|\mathscr{D}|$ and $|\mathscr{E}|$ the same type of restriction as that for $|\mathscr{C}|$ in (17).

3. NONLINEAR INTERFERENCE TERM IN THE DOUBLE-SLIT EXPERIMENT

We apply the results obtained in the preceding section to our problem of calculation of the nonlinear interference term in two dimensions. We consider the case where in Figure 2 the initial condition is such that our wave function $\psi(t=0,A)$ is well localized at $A=S_0$ (source). Various notations to be used below are defined in Figure 2. In the first place we note that the wave function ψ_1 for Figure 1b is given in our approach by

$$\psi_1(T,S_3)=N\cdot K'(t=T,S_3;\,t=0,S_0)_{\mathscr{S}_1} \quad (20)$$

Here the prime on K means that the propagator for (2), which we denote by $K'(b,a)$, is to be used in calculating this quantity, which is a generalization of (11) to the interacting case (2). For paths going from S_0 to S_3 via $\mathscr{S}_1$, it will be a good approximation to replace $K'(b,a)$ by the free propagator $K(b,a)$. Then by making use of (14) and (12) we get

$$\psi_1(T,S_3)\approx N|\mathscr{S}_1|\left(\frac{2\pi i\hbar}{m}\right)^{-3/2}T^{-3/2}l^{-1/2}l_1^{-1/2}\exp\left(i\frac{mL_1^2}{2\hbar T}\right) \quad (21)$$

The wave function corresponding to the paths of the type shown in Figure 2 is now given by

$$\psi_1'(T,S_3)=N\cdot K'(t=T,S_3;\,t=0,S_0)_{\mathscr{S}_1\mathscr{S}_2\mathscr{S}_1} \quad (22)$$

In this case the approximation of the correct propagator $K'(b,a)$ by the free one $K(b,a)$ might not be good. In Figure 4 three types of paths for the free propagator $K(b, a)$ are shown: Types (i) and (iii) represent paths which lie completely on one side of the line AB, while type (ii) represents paths which cross the line AB at least once. The contribution from the type (i) paths is denoted by $K_0(b, a)$, which is clearly equal to the contribution from the type (iii) paths, and that from the type (ii) paths is denoted by $K_1(b, a)$. Then we have

$$K(b, a) = 2K_0(b, a) + K_1(b, a) \tag{23}$$

Suppose that, in the limit where the screen 1 is thin (δ is small), we take $A \in \mathcal{S}_1$ and $B \in \mathcal{S}_2$, that is, the AB in Figure 4 represents a thin screen. Then we would have

$$K'(b, a) \approx 2K_0(b, a) \tag{24}$$

if we take our screen AB to be such that type (ii) paths be forbidden. In this situation the propagator to be used for $\psi_1'(T, S_3)$ in (22) for paths between $\mathcal{S}_1$ and $\mathcal{S}_2$ is $K_0(b, a)$. If we use $K'(b, a)$ of (24) instead of $K_0(b, a)$, the resulting expression for $\psi_1'(T, S_3)$ will also include contributions coming from paths shown in Figure 5.

Now, unfortunately we have no means to calculate $K_0(b, a)$ exactly. We suppose optimistically that in the free propagator $K(b, a)$, (23), the contribution from $K_1(b, a)$ is small compared with that of $K_0(b, a)$, and that $K'(b, a) \sim K(b, a)$ by (23) and (24). [The contribution from the typical path (ii) shown in Figure 4 is the same as that from the path (a) of type (i) in Figure 4, and we can always find shorter paths in category (i). Therefore, considering in the Euclidean space-time, we may say that at least $K_0(b, a)$ is fairly larger than $K_1(b, a)$.] Then we approximate the propagator to be used in (22) by $K(b, a)$. It is assumed that this approximation will retain the essential feature of the wave function $\psi_1'(T, S_3)$.

Then the wave function $\psi_1'(T, S_3)$ coming from paths shown in Figures 2 and 5, which is considered to be a correction to $\psi_1(T, S_3)$, is given by use of (19) to be

$$\psi_1'(T, S_3) \approx N\left(\frac{2\pi i\hbar}{m}\right)^{-5/2} |\mathcal{S}_1|^2 |\mathcal{S}_2| T^{-5/2} l^{-1/2} l_1^{-1/2} \lambda^{-1} L_1^{-1} \exp\left[i\frac{m(L_1+2\lambda)^2}{2\hbar T}\right] \tag{25}$$

We have, therefore, from (21) and (25)

$$\begin{aligned}\psi_1(T, S_3) + \psi_1'(T, S_3) &\equiv \psi_1(T, S_3)[1 + \Delta_1(T, S_3)] \\ &\approx \psi_1(T, S_3)\left\{1 + \left(\frac{2\pi i\hbar}{m}\right)^{-1} |\mathcal{S}_1|\,|\mathcal{S}_2| \right. \\ &\qquad \left. \times T^{-1}\lambda^{-1}L_1^{-1} \exp\left[im\frac{(L_1+2\lambda)^2 - L_1^2}{2\hbar T}\right]\right\}\end{aligned} \tag{26}$$

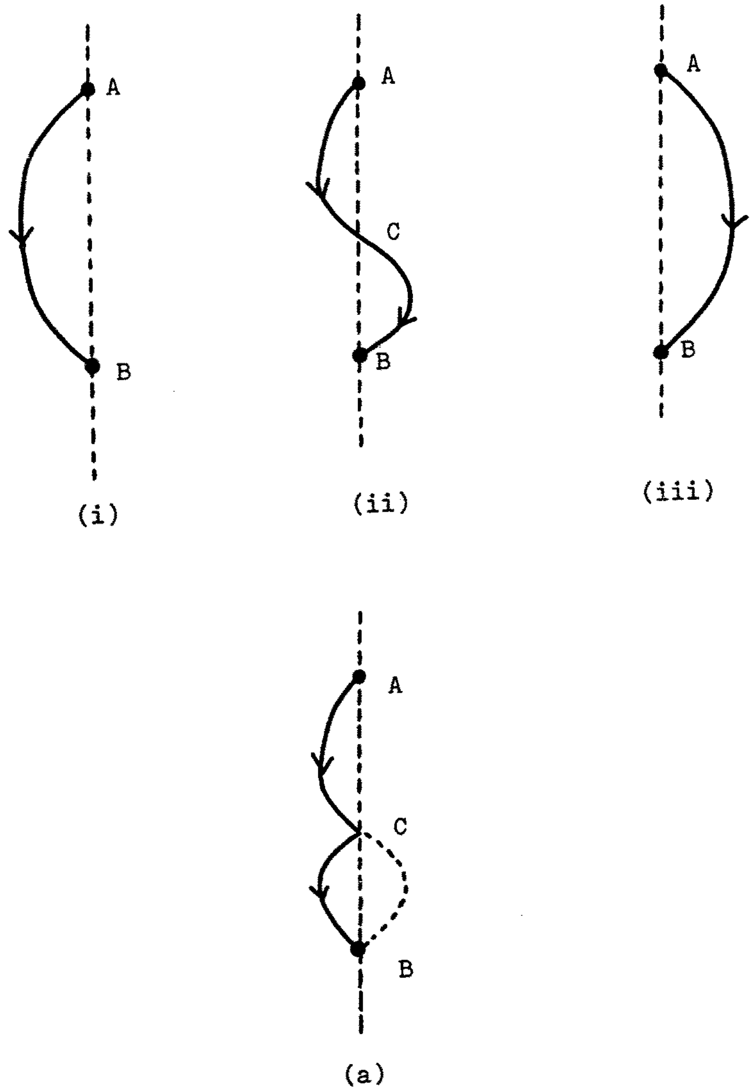

Fig. 4. Three types of paths for the free propagator. The path (a) belonging to type (i) gives the same contribution as that given by the path shown in (ii).

where

$$\Delta_1(T, S_3) \approx \left(\frac{2\pi i\hbar}{m}\right)^{-1} |\mathcal{S}_1|\,|\mathcal{S}_2| T^{-1}\lambda^{-1}L_1^{-1} \exp\left(i\frac{2m\lambda L_1}{\hbar T}\right) \tag{27}$$

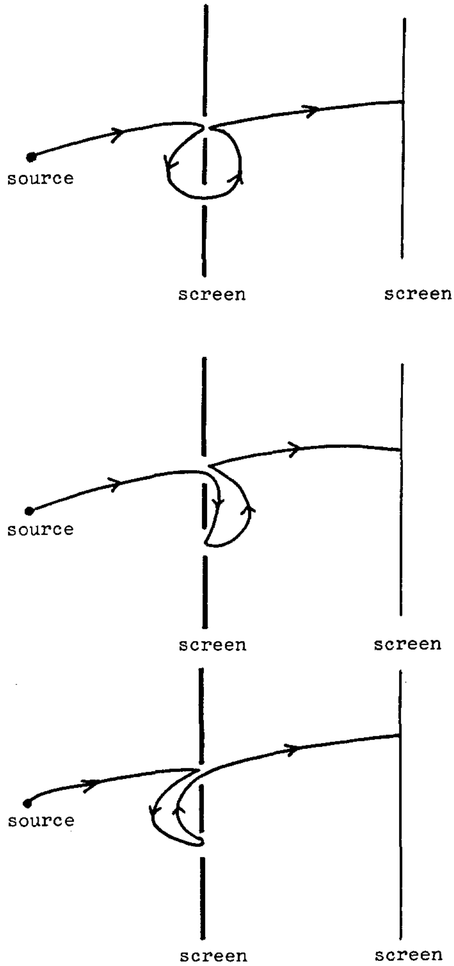

Fig. 5. Three types of paths other than that of Figure 2 passing $\mathscr{S}_1$ twice and $\mathscr{S}_2$ once.

We note that, in spite of the appearance of $\hbar^{-1}$ in (27), the correction term Δ_1 is of order $\hbar$, taking into account (17).

In the same way we obtain for the wave function $\psi_2(T, S_3)$ of Figure 1c and its correction $\psi_2'(T, S_3)$

$$\psi_2(T, S_3) + \psi_2'(T, S_3) = \psi_2(T, S_3)[1 + \Delta_2(T, S_3)]$$

$$\Delta_2(T, S_3) \approx \left(\frac{2\pi i\hbar}{m}\right)^{-1} |\mathscr{S}_1|\,|\mathscr{S}_2| T^{-1}\lambda^{-1}L_2^{-1} \exp\left(i\frac{2m\lambda L_2}{\hbar T}\right) \tag{28}$$

The total probability to find a particle at point S_3 on screen 2 (Figure 2) is

$$\begin{aligned}
&|\psi_1+\psi_1'+\psi_2+\psi_2'|^2\\
&\quad=|\psi_1(1+\Delta_1)+\psi_2(1+\Delta_2)|^2\\
&\quad\approx|\psi_1|^2(1+2\,\mathrm{Re}\,\Delta_1)+|\psi_2|^2(1+2\,\mathrm{Re}\,\Delta_2)+2\,\mathrm{Re}\,(\psi_1^*\psi_2)\\
&\qquad+2\,\mathrm{Re}\,[\psi_1^*\psi_2(\Delta_1^*+\Delta_2)]
\end{aligned}\tag{29}$$

where

$$\begin{aligned}
&\psi_1^*(T,S_3)\psi_2(T,S_3)\\
&\quad\approx|N|^2|\mathscr{S}_1|\,|\mathscr{S}_2|\left(\frac{2\pi\hbar}{m}\right)^{-3}T^{-3}l^{-1}l_1^{-1/2}l_2^{-1/2}\exp\left(im\frac{L_2^2-L_1^2}{2\hbar T}\right)\\
&\quad\approx|N|^2|\mathscr{S}_1|\,|\mathscr{S}_2|\left(\frac{2\pi\hbar}{m}\right)^{-3}T^{-3}l^{-1}l_1^{-1/2}l_2^{-1/2}\exp\left(i\frac{mL\,\Delta L}{\hbar T}\right)
\end{aligned}\tag{30}$$

$$\begin{aligned}
&\Delta_1^*(T,S_3)+\Delta_2(T,S_3)\\
&\approx\left(\frac{2\pi i\hbar}{m}\right)^{-1}|\mathscr{S}_1|\,|\mathscr{S}_2|T^{-1}\lambda^{-1}L^{-1}\left[\exp\left(i\frac{2m\lambda L_2}{\hbar T}\right)-\exp\left(-i\frac{2m\lambda L_1}{\hbar T}\right)\right]\\
&\approx\left(\frac{2\pi i\hbar}{m}\right)^{-1}|\mathscr{S}_1|\,|\mathscr{S}_2|T^{-1}\lambda^{-1}L^{-1}\exp\left(i\frac{m\lambda\,\Delta L}{\hbar T}\right)2i\sin\left(\frac{2m\lambda L}{\hbar T}\right)
\end{aligned}\tag{31}$$

and we have put

$$L=(L_1+L_2)/2,\qquad \Delta L=L_2-L_1\tag{32}$$

The main interference term $2\,\mathrm{Re}\,(\psi_1^*\psi_2)$ is proportional to $\cos(mL\,\Delta L/\hbar T)$, and its correction term is proportional to $\cos(mL\,\Delta L/\hbar T)\sin(2m\lambda L/\hbar T)$. The existence of this correction might be detected experimentally owing to its characteristic dependence on $\lambda L/T$ if one could perform a two-dimensional experiment.

4. DISCUSSIONS

It is straightforward to extend the above analysis to the three-dimensional case with two holes on the screen. The free propagator $K(b,a)$

is now given by

$$K(b,a)=\left[\frac{2\pi i\hbar(t_b-t_a)}{m}\right]^{-3/2}\exp\left[\frac{im\{(x_b-x_a)^2+(y_b-y_a)^2+(z_b-z_a)^2\}}{2\hbar(t_b-t_a)}\right] \tag{33}$$

When we look back at calculations in the appendices, we readily find by following powers of $(2\pi i\hbar/m)$ and doing time integrations that the wave function correction factor Δ_1 is approximately

$$\Delta_1(T,S_3)\approx\left(\frac{2\pi i\hbar}{m}\right)^{-2}T^{-2}\lambda^{-2}|\mathscr{S}_1|\,|\mathscr{S}_2|\exp\left(i\frac{2m\lambda L_1}{\hbar T}\right) \tag{34}$$

where in this case $|\mathscr{S}_i|$ is the volume of the hole i on the screen, and we have the same bound as in (17). [The bound (17) came mainly from the limit of variations along the separation of the two holes, i.e., the y integration. However, we made also the assumption of thin screen, which then restricts the x integration across the screen. Therefore, practically we have another power of $\hbar$ on the right-hand side of (17), and thus $\Delta_1(T,S_3)$ of (34) is expected to be of order $\hbar^2$.]

In this way we see that the interference term has a correction proportional to $\cos(2m\lambda L/\hbar T)$. It will be interesting if experimentally one can detect this small oscillatory behavior of the nonlinear interference term.

It will also be interesting if one can perform time-independent (or time-dependent) perturbation calculation for our problem in the Schrödinger equation. In two-dimensional space we might split the potential $V(x,y)$ in (2) and (3) as

$$V(x,y)=V_0(x,y)+V_1(x,y)+V_2(x,y) \tag{35}$$

where

$$V_0(x,y)=\begin{cases}V & \text{for } 0\le x\le\delta\\ 0 & \text{otherwise}\end{cases} \tag{36}$$

$$V_1(x,y)=\begin{cases}-V & \text{for } (x,y)\in\mathscr{S}_1\\ 0 & \text{otherwise}\end{cases} \tag{37}$$

and

$$V_2(x,y)=\begin{cases}-V & \text{for } (x,y)\in\mathscr{S}_2\\ 0 & \text{otherwise}\end{cases} \tag{38}$$

We then take

$$L_0=\frac{m}{2}(\dot{x}^2+\dot{y}^2)-V_0(x,y) \tag{39}$$

as unperturbed Lagrangian, and expand Green's functions as powers of $V_1(x, y)$ and $V_2(x, y)$. For example, the propagator $K'(b, a)$ of our particle is expanded as

$$K'(b, a) = K_0(b, a) - \frac{i}{\hbar} \int K_0(b, c)\{V_1(c) + V_2(c)\} K_0(c, a)\, dx_c\, dy_c\, dt_c$$

$$+ \left(-\frac{i}{\hbar}\right)^2 \int\int K_0(b, c)[V_1(c) + V_2(c)] K_0(c, d)[V_1(d) + V_2(d)]$$

$$\times K_0(d, a)\, dx_c\, dy_c\, dt_c\, dx_d\, dy_d\, dt_d + \cdots \tag{40}$$

where $K_0(b, a)$ here corresponds to L_0 of (39). Then the wave function ψ_1 appears as of order V_1, and ψ_1' as of order $V_1^2 V_2$. For this purpose we have to calculate tunneling amplitudes for the Lagrangian L_0. We have not yet pursued this investigation to produce concrete results.

APPENDIX 1: ESTIMATION OF THE INTEGRAL (12)

We are going to calculate the integral

$$\begin{aligned} I_1 &= \frac{1}{t_b - t_a} \int_{t_a}^{t_b} dt_c\, K(b, c) K(c, a) \\ &= \left(\frac{2\pi i\hbar}{m}\right)^{-2} \frac{1}{T} \int_0^T \frac{dt}{t(T-t)} \exp\left[i \frac{m}{2\hbar}\left(\frac{l_{ca}^2}{t} + \frac{l_{bc}^2}{T-t}\right)\right] \end{aligned} \tag{A1}$$

where we have put

$$\begin{aligned} T &= t_b - t_a \\ t &= t_c - t_a \\ l_{ca}^2 &= (x_c - x_a)^2 + (y_c - y_a)^2 \\ l_{bc}^2 &= (x_b - x_c)^2 + (y_b - y_c)^2 \end{aligned} \tag{A2}$$

Here we note that the integral in (A1) is not well defined for $l_{ca} = 0$ or $l_{bc} = 0$. When $l_{ca} \neq 0$ and $l_{bc} \neq 0$, we expect rapid oscillations of the phase of the integrand for $t = 0$ and $t = T$ providing us with a converging expression.

By changing variables, we rewrite I_1 as

$$I_1 = \left(\frac{2\pi i\hbar}{m}\right)^{-2} \frac{2}{T^2} e^{i\xi} \int_{-1}^{1} (1-z^2)^{-1} \exp\left[i\xi \frac{(z-z_0)^2}{1-z^2} \right] dz \tag{A3}$$

where

$$z = 1 - \frac{2t}{T}$$

$$\xi = \frac{m(l_{ca}+l_{bc})^2}{2\hbar T} \tag{A4}$$

$$z_0 = \frac{l_{bc}-l_{ca}}{l_{bc}+l_{ca}}$$

Since the integral in (A3) is dominated by the contribution from the integrand around $z \approx z_0$, we approximate as follows:

$$\begin{aligned} I_1 &\approx \left(\frac{2\pi i\hbar}{m}\right)^{-2} \frac{2}{T^2} e^{i\xi} \int_{-1}^{1} (1-z_0^2)^{-1} \exp\left[i\xi \frac{(z-z_0)^2}{1-z_0^2} \right] dz \\ &\approx \left(\frac{2\pi i\hbar}{m}\right)^{-2} \frac{2}{T^2} e^{i\xi}(1-z_0^2)^{-1} \int_{-\infty}^{\infty} \exp\left[i\xi \frac{(z-z_0)^2}{1-z_0^2} \right] dz \\ &= \left(\frac{2\pi i\hbar}{m}\right)^{-2} \frac{2}{T^2} e^{i\xi}(1-z_0^2)^{-1} \left[\frac{\pi i(1-z_0^2)}{\xi} \right]^{1/2} \\ &= \left(\frac{2\pi i\hbar}{m}\right)^{-3/2} T^{-3/2} l_{ca}^{-1/2} l_{bc}^{-1/2} e^{i\xi} \end{aligned} \tag{A5}$$

This estimate will be correct up to a factor of order 1 for ξ not too small, and the varying phase of I_1 will be mainly given by $\exp(i\xi)$ of (A5).

APPENDIX 2: ESTIMATION OF THE INTEGRAL IN (18)

We calculate the integral appearing in (18) in a similar way as in Appendix 1:

$$I_2 = \frac{1}{t_b - t_a} \int_{t_a}^{t_b} dt_d\, K(b,d)K(d,a)_{\mathscr{C}}$$

$$
\begin{aligned}
&= \left(\frac{2\pi i\hbar}{m}\right)^{-5/2} |\mathscr{C}| l_{dc}^{-1/2} l_{ca}^{-1/2} \frac{1}{T} \int_0^T dt\, t^{-3/2}(T-t)^{-1} \\
&\quad \times \exp\left\{ i \frac{m}{2\hbar} \left[\frac{(l_{dc}+l_{ca})^2}{t} + \frac{l_{bd}^2}{T-t} \right] \right\} \\
&= \left(\frac{2\pi i\hbar}{m}\right)^{-5/2} |\mathscr{C}| T^{-5/2} l_{dc}^{-1/2} l_{ca}^{-1/2} 2^{3/2} e^{i\zeta} \int_{-1}^{1} dz (1-z)^{-3/2}(1+z)^{-1} \\
&\quad \times \exp\left[i\zeta \frac{(z-z_0)^2}{1-z^2} \right]
\end{aligned}
\tag{A6}
$$

where

$$
\begin{aligned}
T &= t_b - t_a \\
t &= t_d - t_a \\
z_0 &= \frac{l_{bd} - l_{dc} - l_{ca}}{l_{dc} + l_{ca} + l_{bd}} \\
\zeta &= \frac{m(l_{dc} + l_{ca} + l_{bd})^2}{2\hbar T}
\end{aligned}
\tag{A7}
$$

and other notations will be obvious. With the same type of approximation as in Appendix 1 we get

$$
\begin{aligned}
I_2 &\approx \left(\frac{2\pi i\hbar}{m}\right)^{-5/2} |\mathscr{C}| T^{-5/2} l_{dc}^{-1/2} l_{ca}^{-1/2} 2^{3/2} e^{i\zeta} (1-z_0)^{-3/2}(1+z_0)^{-1} \left[\frac{\pi i (1-z_0^2)}{\zeta} \right]^{1/2} \\
&\approx \left(\frac{2\pi i\hbar}{m}\right)^{-2} |\mathscr{C}| T^{-2} l_{dc}^{-1/2} l_{ca}^{-1/2} l_{bd}^{-1/2} (l_{bd} + l_{dc} + l_{ca})^{-1/2} e^{i\zeta}
\end{aligned}
\tag{A8}
$$

where in the last line we have replaced $(l_{bd} + l_{dc} + l_{ca})/(l_{dc} + l_{ca})$ by 1, since our estimate is good at best up to a factor of order 1.

APPENDIX 3: ESTIMATION OF THE INTEGRAL IN (19)

We calculate the integral appearing in (19) in a similar way as in Appendices 1 and 2:

$$
I_3 = \frac{1}{t_b - t_a} \int_{t_a}^{t_b} dt_e\, K(b, e) K(e, a)_{\mathscr{D}\mathscr{C}}
$$

$$= \left(\frac{2\pi i\hbar}{m}\right)^{-3} |\mathscr{C}|\,|\mathscr{D}|\, T^{-1} l_{ed}^{-1/2} l_{dc}^{-1/2} l_{ca}^{-1/2} (l_{ed}+l_{dc}+l_{ca})^{-1/2}$$

$$\times \int_0^T dt\, t^{-2}(T-t)^{-1} \exp\left\{ i\frac{m}{2\hbar}\left[\frac{(l_{ed}+l_{dc}+l_{ca})^2}{t} + \frac{l_{be}^2}{T-t}\right]\right\}$$

$$= \left(\frac{2\pi i\hbar}{m}\right)^{-3} |\mathscr{C}|\,|\mathscr{D}| 2^2 T^{-3} l_{ed}^{-1/2} l_{dc}^{-1/2} l_{ca}^{-1/2} (l_{ed}+l_{dc}+l_{ca})^{-1/2} e^{i\eta}$$

$$\times \int_{-1}^{1} dz (1-z)^{-2}(1+z)^{-1} \exp\left[i\eta \frac{(z-z_0)^2}{1-z^2}\right]$$

$$\approx \left(\frac{2\pi i\hbar}{m}\right)^{-3} |\mathscr{C}|\,|\mathscr{D}| 2^2 T^{-3} l_{ed}^{-1/2} l_{dc}^{-1/2} l_{ca}^{-1/2} (l_{ed}+d_{dc}+l_{ca})^{-1/2} e^{i\eta}$$

$$\times (1-z_0)^{-2}(1+z_0)^{-1}\left[\frac{\pi i(1-z_0^2)}{\eta}\right]^{1/2}$$

$$\approx \left(\frac{2\pi i\hbar}{m}\right)^{-5/2} |\mathscr{C}|\,|\mathscr{D}|\, T^{-5/2} l_{be}^{-1/2} l_{ed}^{-1/2} l_{dc}^{-1/2} l_{ca}^{-1/2} (l_{be}+l_{ed}+l_{dc}+l_{ca})^{-1} e^{i\eta} \quad \text{(A9)}$$

where

$$\begin{aligned} T &= t_b - t_a \\ t &= t_e - t_a \\ z_0 &= \frac{l_{be}-l_{ed}-l_{dc}-l_{ca}}{l_{be}+l_{ed}+l_{dc}+l_{ca}} \\ \eta &= \frac{m(l_{be}+l_{ed}+l_{dc}+l_{ca})^2}{2\hbar T} \end{aligned} \quad \text{(A10)}$$

and in the last line in (A9) we have replaced

$$(l_{be}+l_{ed}+l_{dc}+l_{ca})/(l_{ed}+l_{dc}+l_{ca})$$

by 1 in the same spirit as in Appendix 2.

REFERENCE

Feynman, R. P., and Hibbs, A. R. (1965). *Quantum Mechanics and Path Integrals.* McGraw-Hill, New York.